From Finite-System Entropy to Entropy Rate for a Hidden Markov Process

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***Abstract*—A recent result presented the expansion for the entropy rate of a hidden Markov process (HMP) as a power series in the noise variable ϵ. The coefficients of the expansion around the noiseless ($\epsilon = 0$) limit were calculated up to 11th order, using a conjecture that relates the entropy rate of an HMP to the entropy of a process of finite length (which is calculated analytically). In this letter, we generalize and prove the conjecture and discuss its theoretical and practical consequences.**

***Index Terms*—Entropy, hidden Markov process (HMP), Taylor series.**

I. INTRODUCTION

LET $\{X_N\}$ be a finite state stationary Markov process over the alphabet $\Sigma = \{1, \ldots, s\}$, and let $\{Y_N\}$ be its noisy observation (on the same alphabet). The process Y is generated by the Markov transition matrix $M = M_{s\times s} = \{m_{ij}\}$ and the emission matrix $I + \epsilon T$, where I is the $s \times s$ identity matrix, the matrix $T = T_{s\times s} = \{t_{ij}\}$ satisfies $t_{ii} < 0, t_{ij} \geq 0, \forall i \neq j$, and $\sum_{j=1}^{s} t_{ij} = 0$, and $\epsilon > 0$ is some constant. (There is no loss of generality here, as any stochastic matrix can be represented as $I + \epsilon T$.) This yields the probabilities $P(X_{N+1} = j | X_N = i) = m_{ij}$ and $P(Y_N = j | X_N = i) = \delta_{ij} + \epsilon t_{ij}$, where δ is Kronecker's delta. We consider the case of high signal to noise ratio ("High-SNR"), characterized by small values of ϵ, and assume strictly positive M ($m_{ij} > 0$) with a unique stationary distribution.

The process Y can be viewed as an observation of X through a noisy channel. It is a *hidden Markov process (HMP)*, governed by the parameters M, T, and ϵ. *HMPs* have a rich theory, with applications in various fields, such as speech recognition [1], information theory [2], and signal processing [3]. While we concentrate on a finite-state first-order *HMP*, our results can be easily generalized to more cases (e.g., continuous observations).

An important quantity for a stochastic process is the Shannon entropy rate, which measures its "uncertainty per-symbol" [4]. More formally, for $i \leq j$, let $[Y]_i^j$ denote the vector $(Y_i, \ldots, Y_j)$. The entropy rate of Y is defined as

$$\bar{H}(Y) = \lim_{N\to\infty} \frac{H\left([Y]_1^N\right)}{N} \tag{1}$$

where $H(Y) = -\sum_Y P(Y) \log P(Y)$; sometimes we omit the realization y of the variable Y, so $P(Y)$ should be understood as $P(Y = y)$. For a finite-entropy stationary process, the limit (1) exists, and $\bar{H}$ can also be computed via the conditional entropy [5] as $\bar{H}(Y) = \lim_{N\to\infty} H(Y_N | [Y]_1^{N-1})$. Here, $H(U|V)$ represents the conditional entropy, which for random variables U and V is the average uncertainty of the conditional distribution of U given V, that is, $H(U|V) = \sum_v P(V = v) H(U|V = v)$. By the entropy chain rule, it is also given as a difference of entropies, $H(U|V) = H(U, V) - H(V)$. This relation will be used below.

There is at present no explicit expression for the entropy rate of an *HMP* [2], [6]. Few recent works [6]–[8] have studied the asymptotic behavior of $\bar{H}$ in several regimes, albeit giving rigorously only bounds or at most second-order [8] behavior. Here, we generalize and prove a relationship, first posed in [8] as a conjecture, thereby turning the computation presented there, of $\bar{H}$ as a series expansion up to 11th order in ϵ, into a rigorous statement.

II. THEOREM STATEMENT AND PROOF

We first state our main result, which will be proven at the end of the section.

Theorem 1: Let $H_N \equiv H_N(M, T, \epsilon) = H([Y]_1^N)$ be the entropy of a system of length N, and let $C_N = H_N - H_{N-1}$. Let $B_\rho(0) \subset \mathbb{C}$ be some (complex) neighborhood of zero, in which the functions $\{C_N\}$ and $\bar{H}$ are analytic in ϵ, with Taylor expansions given by

$$C_N(M, T, \epsilon) = \sum_{k=0}^{\infty} C_N^{(k)} \epsilon^k, \quad \bar{H}(M, T, \epsilon) = \sum_{k=0}^{\infty} C^{(k)} \epsilon^k. \tag{2}$$

The coefficients $C_N^{(k)}$ are functions of M and T. From now on, we omit this dependence. Then

$$N \geq \left\lceil \frac{k+3}{2} \right\rceil \Rightarrow C_N^{(k)} = C^{(k)}. \tag{3}$$

Analyticity of $\{C_N\}$ and $\bar{H}$ around $\epsilon = 0$ was recently shown in [9]. One may also use [10], which showed that the law of the

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process Y is Gibbsian, together with the complete analyticity results for Gibbsian measures of [11], to deduce analyticity of $\bar{H}$. C_N is in fact an upper-bound [5] for $\bar{H}$. The behavior stated in Theorem 1 was discovered using symbolic computations but was proven only for $k \leq 2$, in the binary symmetric case [8]. Although it may appear technically involved, our proof is based on two simple ideas.

First, we distinguish between the noise parameters at different sites. We thus consider a more general process $\{Z_N\}$, where Z_i's emission matrix is $I + \epsilon_i T$. The process $\{Z_N\}$ is determined by M, T, and $[\epsilon]_1^N$. We define the following functions:

$$F_N\left(M,T,[\epsilon]_1^N\right) = H\left([Z]_1^N\right) - H\left([Z]_1^{N-1}\right). \tag{4}$$

Setting all the ϵ_i's equal reduces this to the Y process, and in particular, $F_N(M,T,(\epsilon,\ldots,\epsilon)) = C_N(\epsilon)$.

Second, we observe that if a particular ϵ_i is set to zero, we must have $Z_i = X_i$. Thus, conditioning back to the past is "blocked." This is used to prove the following.

Lemma 1: If $\epsilon_j = 0$ for some $1 < j < N$, then

$$F_N\left([\epsilon]_1^N\right) = F_{N-j+1}\left([\epsilon]_j^N\right). \tag{5}$$

Proof: F can be written as the sum

$$F_N = -\sum_{[Z]_1^N}\left[P\left([Z]_1^{N-1}\right)P\left(Z_N|[Z]_1^{N-1}\right)\right.$$
$$\left.\times \log P\left(Z_N|[Z]_1^{N-1}\right)\right]. \tag{6}$$

The dependence on $[\epsilon]_1^N$ and M,T is hidden in the probabilities $P(\ldots)$. Since $\epsilon_j = 0$, we have $X_j = Z_j$, and conditioning further to the past is "blocked"

$$\epsilon_j = 0 \Rightarrow P\left(Z_N|[Z]_1^{N-1}\right) = P\left(Z_N|[Z]_j^{N-1}\right). \tag{7}$$

Substituting in (6) gives

$$\begin{aligned} F_N &= -\sum_{[Z]_1^N}\left[P\left([Z]_1^{N-1}\right)P\left(Z_N|[Z]_j^{N-1}\right)\right.\\ &\qquad\left.\times \log P\left(Z_N|[Z]_j^{N-1}\right)\right]\\ &= -\sum_{[Z]_j^N} P\left([Z]_j^N\right)\log P\left(Z_N|[Z]_j^{N-1}\right)\\ &= F_{N-j+1}. \end{aligned} \tag{8}$$

■

Let $\vec{k} = [k]_1^N$ be a vector with $k_i \in \{\mathbb{N} \cup 0\}$. Define its "weight" as $\omega(\vec{k}) = \sum_{i=1}^N k_i$. Define also

$$F_N^{\vec{k}} \equiv \left.\frac{\partial^{\omega(\vec{k})}F_N}{\partial\epsilon_1^{k_1},\ldots,\partial\epsilon_N^{k_N}}\right|_{\vec{\epsilon}=0}. \tag{9}$$

$C_N^{(k)}$ is obtained by summing the contributions $F_N^{\vec{k}}$ of all the vectors $\vec{k}$'s with weight k

$$C_N^{(k)} = \frac{1}{k!}\sum_{\vec{k},\omega(\vec{k})=k} F_N^{\vec{k}}. \tag{10}$$

The next lemma shows that many such $\vec{k}$'s give zero contribution to the sum.

Lemma 2: Let $\vec{k} = [k]_1^N$. If $\exists i,j, 1 \leq i < j < N$, with $k_i \geq 1$, $k_j \leq 1$, then $F_N^{\vec{k}} = 0$.

Proof: Assume first $k_j = 0$. Using lemma 1, we get

$$\begin{aligned} F_N^{\vec{k}} &\equiv \left.\frac{\partial^{\omega(\vec{k})}F_N\left([\epsilon]_1^N\right)}{\partial\epsilon_1^{k_1},\ldots,\partial\epsilon_N^{k_N}}\right|_{\vec{\epsilon}=0} = \left.\frac{\partial^{\omega(\vec{k})}F_{N-j+1}\left([\epsilon]_j^N\right)}{\partial\epsilon_1^{k_1},\ldots,\partial\epsilon_N^{k_N}}\right|_{\vec{\epsilon}=0}\\ &= \left.\frac{\partial^{\omega(\vec{k})-1}}{\partial\epsilon_1^{k_1},\ldots,\partial\epsilon_i^{k_i-1},\ldots,\partial\epsilon_N^{k_N}}\left[\frac{\partial F_{N-j+1}\left([\epsilon]_j^N\right)}{\partial\epsilon_i}\right]\right|_{\vec{\epsilon}=0}\\ &= 0. \end{aligned} \tag{11}$$

Assume now $k_j = 1$. Write the probability of Z

$$\begin{aligned} P\left([Z]_1^N\right) &= \sum_{[X]_1^N} P\left([X]_1^N\right)P\left([Z]_1^N|[X]_1^N\right)\\ &= \sum_{[X]_1^N} P\left([X]_1^N\right)\prod_{i=1}^N\left(\delta_{X_iZ_i} + \epsilon_i t_{X_iZ_i}\right). \end{aligned} \tag{12}$$

Let $[Z]_1^{N^{(j\to a)}}$ denote the vector we get from $[Z]_1^N$ by changing Z_j to a (while keeping other coordinates). Differentiating with respect to ϵ_j gives (see [12] for more details)

$$\begin{aligned} &\left.\frac{\partial P\left([Z]_1^N\right)}{\partial\epsilon_j}\right|_{\epsilon_j=0}\\ &= \left.\sum_{[X]_1^N}\left[P\left([X]_1^N\right)t_{X_jZ_j}\prod_{i\neq j}\left(\delta_{X_iZ_i} + \epsilon_i t_{X_iZ_i}\right)\right]\right|_{\epsilon_j=0}\\ &= \left.\left\{\sum_{a=1}^s t_{aZ_j}P\left([Z]_1^{N^{(j\to a)}}\right)\right\}\right|_{\epsilon_j=0}. \end{aligned} \tag{13}$$

By Bayes' rule $P(Z_N|[Z]_1^{N-1}) = P([Z]_1^N)/P([Z]_1^{N-1})$, we get

$$\begin{aligned} &\left.\frac{\partial P\left(Z_N|[Z]_1^{N-1}\right)}{\partial\epsilon_j}\right|_{\epsilon_j=0} = \frac{1}{P\left([Z]_1^{N-1}\right)}\sum_{a=1}^s t_{aZ_j}\\ &\times\left.\left[P\left([Z]_1^{N^{(j\to a)}}\right) - P\left(Z_N|[Z]_1^{N-1}\right)P\left([Z]_1^{N-1^{(j\to a)}}\right)\right]\right|_{\epsilon_j=0}. \end{aligned} \tag{14}$$

This gives

$$\left.\frac{\partial\left[P\left([Z]_1^N\right)\log P\left(Z_N|[Z]_1^{N-1}\right)\right]}{\partial\epsilon_j}\right|_{\epsilon_j=0} = \sum_{a=1}^{s} t_{aZ_j}\left\{P\left([Z]_1^{N(j\to a)}\right)\log P\left(Z_N|[Z]_1^{N-1}\right) + P\left([Z]_1^{N(j\to a)}\right) - P\left(Z_N|[Z]_1^{N-1}\right) \times P\left([Z]_1^{N-1(j\to a)}\right)\right\}\Big|_{\epsilon_j=0} \quad (15)$$

and therefore

$$\begin{aligned}\left.\frac{\partial F_N}{\partial\epsilon_j}\right|_{\epsilon_j=0} &= -\sum_{a=1}^{s} t_{aZ_j} \times\left\{\sum_{[Z]_1^N}\left[P\left([Z]_1^{N(j\to a)}\right)\log P\left(Z_N|[Z]_1^{N-1}\right) - P\left(Z_N|[Z]_1^{N-1}\right)\times P\left([Z]_1^{N-1(j\to a)}\right)\right]\right\}\Bigg|_{\epsilon_j=0}\\ &= \left\{-\sum_{a=1}^{s} t_{aZ_j}\times\sum_{[Z]_j^N}\left[P\left([Z]_j^{N(1\to a)}\right)\log P\left(Z_N|[Z]_j^{N-1}\right) - P\left(Z_N|[Z]_j^{N-1}\right)\times P\left([Z]_j^{N-1(1\to a)}\right)\right]\right\}\Bigg|_{\epsilon_1=0}. \quad (16)\end{aligned}$$

The latter equality comes from using (7), which "blocks" the dependence backward. Equation (16) shows that ϵ_i does not appear in $(\partial F_N/\partial\epsilon_j)|_{\epsilon_j=0}$ for $i < j$; therefore, $(\partial^{k_i+1}F_N/\partial\epsilon_i^{k_i}\partial\epsilon_j)|_{\epsilon_j=0} = 0$ and $F_N^{\vec{k}} = 0$. ■

Before proving Theorem 1, we show here that adding zeros to the left of $\vec{k}$ leaves $F_N^{\vec{k}}$ unchanged.

Lemma 3: Let $\vec{k} = [k]_1^N$ with $k_1 \le 1$. Denote $\vec{k}^{(r)}$ the concatenation of $\vec{k}$ and r zeros to the left: $\vec{k}^{(r)} = (\underbrace{0,\ldots,0}_{r}, k_1,\ldots,k_N)$. Then

$$F_N^{\vec{k}} = F_{r+N}^{\vec{k}^{(r)}}, \quad \forall r\in\mathbb{N}. \quad (17)$$

Proof: Assume first $k_1 = 0$. Using lemma 1, we get

$$\begin{aligned}F_{r+N}^{\vec{k}^{(r)}}\left([\epsilon]_1^{r+N}\right) &= \left.\frac{\partial^{\omega(\vec{k}^{(r)})}F_{r+N}\left([\epsilon]_1^{r+N}\right)}{\partial\epsilon_{r+2}^{k_2},\ldots,\partial\epsilon_{r+N}^{k_N}}\right|_{\vec{\epsilon}=0}\\ &= \left.\frac{\partial^{\omega(\vec{k})}F_N\left([\epsilon]_{r+1}^{r+N}\right)}{\partial\epsilon_{r+2}^{k_2},\ldots,\partial\epsilon_{r+N}^{k_N}}\right|_{\vec{\epsilon}=0}\\ &= F_N^{\vec{k}}\left([\epsilon]_{r+1}^{r+N}\right). \quad (18)\end{aligned}$$

The case $k_1 = 1$ is reduced back to the case $k_1 = 0$ by taking the derivative. Using (16) and (18), we get

$$\begin{aligned}F_{N+1}^{\vec{k}^{(1)}}\left([\epsilon]_1^{N+1}\right) &= \frac{\partial^{\omega(\vec{k})-1}}{\partial\epsilon_3^{k_2}\ldots\partial\epsilon_{N+1}^{k_N}}\left[\left.\frac{\partial F_{N+1}}{\partial\epsilon_2}\right|_{\epsilon_2=0}\right]\Bigg|_{\vec{\epsilon}=0}\\ &= \frac{\partial^{\omega(\vec{k})-1}}{\partial\epsilon_3^{k_2}\ldots\partial\epsilon_{N+1}^{k_N}} \times\left\{-\sum_{a=1}^{s}t_{aZ_2}\times\sum_{[Z]_1^{N+1}}\left[P\left([Z]_1^{N+1(2\to a)}\right)\log P\left(Z_{N+1}|[Z]_1^N\right) - P\left(Z_{N+1}|[Z]_1^N\right)\times P\left([Z]_1^{N(2\to a)}\right)\right]\Big|_{\epsilon_2=0}\right\}\Bigg|_{[\epsilon]_1^{N+1}=0}\\ &= \frac{\partial^{\omega(\vec{k})-1}}{\partial\epsilon_2^{k_2}\ldots\partial\epsilon_N^{k_N}} \times\left\{-\sum_{a=1}^{s}t_{aZ_2}\times\sum_{[Z]_1^{N}}\left[P\left([Z]_1^{N(1\to a)}\right)\log P\left(Z_{N}|[Z]_1^{N-1}\right) - P\left(Z_{N}|[Z]_1^{N-1}\right)\times P\left([Z]_1^{N(1\to a)}\right)\right]\Big|_{\epsilon_1=0}\right\}\Bigg|_{[\epsilon]_1^{N}=0}\\ &= F_N^{\vec{k}}\left([\epsilon]_1^N\right). \quad (19)\end{aligned}$$

This proved the claim for $r = 1$. The claim for larger r's follows by induction. ■

We are now ready to prove our main theorem, which follows directly from lemmas 2 and 3.

Proof (Theorem 1): Let $\vec{k} = [k]_1^N$ with $\omega(\vec{k}) = k$. Define its "length" as $l(\vec{k}) = N + 1 - \min_{k_i>1}\{i\}$. It easily follows from lemma 2 that $F_N^{\vec{k}} \ne 0 \Rightarrow l(\vec{k}) \le \lceil (k+3)/2\rceil - 1$. Thus, according to lemma 3, we have

$$F_N^{\vec{k}} = F_{\lceil\frac{k+3}{2}\rceil}^{\left(k_{N-\lceil\frac{k+3}{2}\rceil+1},\ldots,k_N\right)} \quad (20)$$

for all $\vec{k}$'s in the sum. Summing over all $F_N^{\vec{k}}$ with the same "weight" gives $C_N^{(k)} = C_{\lceil(k+3)/2\rceil}^{(k)}, \forall N > \lceil(k+3)/2\rceil$. However, from the analyticity of C_N and $\bar{H}$ near $\epsilon = 0$, it can be shown by induction that $\lim_{N\to\infty}C_N^{(k)} = C^{(k)}$; therefore, $C_N^{(k)} = C^{(k)}, \forall N \ge \lceil(k+3)/2\rceil$. ■

III. Conclusion

The theorem proven above sheds light on the connection between finite and infinite chains and gives a practical and straightforward way to compute the entropy rate as a series expansion in ϵ up to an arbitrary power. The surprising "settling" of the expansion coefficients $C_N^{(k)} = C^{(k)}$ for $N \geq \lceil (k+3)/2 \rceil$ holds for the entropy. For other functions involving only conditional probabilities (e.g., relative entropy between two *HMPs*), a weaker result holds: the coefficients "settle" for $N \geq k+2$. One can expand the entropy rate in several parameter regimes. As it turns out, exactly the same "settling" as was proven in Theorem 1 happens in the "almost memoryless" regime, where the transition matrix M is close to a matrix, which makes the X_i's i.i.d, (i.e., a matrix whose rows are identical). This and other regimes, as well as the analytic behavior of the *HMP* [9], will be discussed elsewhere.

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